ON SPECIAL RATIONAL CURVES IN GRASSMANNIANS.

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ABSTRACT. We characterize, among all morphisms $\mathbb{P}^1 \to \mathbb{G}(d, 2d)$, those which are $\mathrm{GL}_{2d}(\mathbb{C})$-equivalent to the canonical morphism induced by the Morita equivalence $\mathbb{C}^d \otimes_{\mathbb{C}} -$.

1. Introduction. Understanding curves, especially rational, in Grassmann varieties is very helpful in many problems in Algebraic Geometry [3], [11], [15], [16], [18], Algebraic Topology [9], [10], [12], and Control Theory [1], [2], [4], [5], [7], [8], [13], [19], [20].

In this article we are interested in the characterization of curves in Grassmann varieties arising as a result of the following canonical construction.

The Morita equivalence of $\mathbb{C}$-modules and $\mathrm{M}_d(\mathbb{C})$-modules (where $\mathrm{M}_d(\mathbb{C})$ denotes the ring of $d \times d$ matrices with complex entries) is defined by sending a $\mathbb{C}$-module M to $\mathbb{C}^d \otimes_{\mathbb{C}} M$. This identifies grassmannians of $\mathbb{C}$-sub-modules of the $\mathbb{C}$-module $\mathbb{C}^d$ isomorphic to $\mathbb{C}^k$ with grassmannians of $\mathrm{M}_d(\mathbb{C})$-sub-modules of the $\mathrm{M}_d(\mathbb{C})$-module $\mathbb{C}^{dn} = \mathbb{C}^d \otimes_{\mathbb{C}} \mathbb{C}^n$ isomorphic to $\mathbb{C}^{dk} = \mathbb{C}^d \otimes_{\mathbb{C}} \mathbb{C}^k$

$$\mathbb{G}(k, n) \xrightarrow{\cong} \mathbb{G}_{M_d}(dk, dn).$$

The faithfull forgetting functor $\mathrm{M}_d(\mathbb{C}) - \mathrm{mod} \to \mathbb{C} - \mathrm{mod}$ defines a canonical closed embedding

$$\mathbb{G}_{M_d}(dk, dn) \hookrightarrow \mathbb{G}(dk, dn).$$

Definition. We call the composition $\mathbb{G}(k, n) \to \mathbb{G}(dk, dn)$ of the above morphisms the *Morita morphism*. For $(k, n) = (1, 2)$ the above composition is a smooth rational curve

$$f_M : \mathbb{P}^1 \to \mathbb{G}(d, 2d),$$

which we call the *Morita curve*.

Its geometry can be described as follows. Let

(1) $$0 \to S \to \mathbb{C}^{2d} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{G}} \to Q \to 0.$$

be the tautological short exact sequence on the grassmannian $\mathbb{G} = \mathbb{G}(d, 2d)$, where $\mathcal{O}_{\mathbb{G}}$ denotes the sheaf of holomorphic functions and S (resp. Q) denotes the locally

free sheaf of holomorphic sections of the tautological sub-bundle (resp. quotient bundle) of the trivial bundle. The fiber S_x of the tautological sub-bundle S at a point $x \in \mathbb{G}$ corresponding to a subspace of $\mathbb{C}^{2d}$ can be canonically identified with this subspace. For $d = 1$ the short exact sequence (1) becomes the twisted Euler sequence

$$(2) \qquad 0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{C}^2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1} \to \det_{\mathbb{C}}(\mathbb{C}^2) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}(1) \to 0,$$

where $\det_{\mathbb{C}}(\mathbb{C}^2) = \bigwedge_{\mathbb{C}}^2 \mathbb{C}^2$ is a one dimensional space transforming under a linear change of the base vectors in $\mathbb{C}^2$ by the determinant of the transition matrix, and $\mathcal{O}_{\mathbb{P}^1}(1)$ is the dual of the invertible tautological sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)$. .... The sheaf of holomorphic 1-forms $\Omega_{\mathbb{G}}^1$ can be expressed as $\Omega_{\mathbb{G}}^1 = S \otimes Q^{\vee}$. Since $\Omega_{\mathbb{P}^1}^1 = \det_{\mathbb{C}}(\mathbb{C}^2)^{-1} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}(-2)$ the differential $df : f^* \Omega_{\mathbb{G}}^1 \to \Omega_{\mathbb{P}^1}^1$ of any rational curve $f : \mathbb{P}^1 \to \mathbb{G}$ is equivalent to a morphism of locally free sheaves

$$(3) \qquad f^* S \to \det_{\mathbb{C}}(\mathbb{C}^2)^{-1} \otimes_{\mathbb{C}} f^* Q(-2).$$

By (1) the determinant of (3) is equivalent to an element

$$(4) \qquad \Delta(f) \in \det_{\mathbb{C}}(\mathbb{C}^d)^2 \otimes_{\mathbb{C}} \mathrm{H}^0(\mathbb{P}^1, (f^*(\det S)(d))^{-2}).$$

The Plücker embedding $\mathbb{G} \hookrightarrow \mathbb{P}^N$, $N = \binom{2d}{d} - 1$, is defined by means of the very ample line bundle $\mathcal{L} = (\det S)^{-1}$ and defines the *Plücker degree* of a morphism f

$$(5) \qquad \deg(f) := -\deg f^* \det S.$$

One has the Grothendieck splitting

$$(6) \qquad f^* S \overset{\cong}{\to} \bigoplus_{i=1}^{d} \mathcal{O}_{\mathbb{P}^1}(-a_i).$$

We can assume that

$$(7) \qquad 0 \le a_1 \le \ldots \le a_d,$$

because $S^{\vee}$ is globally generated. We will denote by $\varpi(f)$ the *width of the splitting* (6), i.e.

$$(8) \qquad \varpi(f) = a_d - a_1.$$

In the case of the Morita curve f_M the morphism (3) is an isomorphism and $f^* S = \mathbb{C}^d \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore

$$(9) \qquad \deg(f_M) = d, \quad \Delta(f_M) \ne 0, \quad \varpi(f_M) = 0.$$

Note that every f which is $\mathrm{GL}_{2d}(\mathbb{C})$-equivalent to f_M satisfies (9) as well. We prove the following theorem

Theorem. *Let* $f : \mathbb{P}^1 \to \mathbb{G}(d, 2d)$ *be a rational curve such that*

(10) $$\deg(f) = d, \quad \Delta(f) \neq 0, \quad \varpi(f) \leq 3.$$

Then f *is* $\mathrm{GL}_{2d}(\mathbb{C})$*-equivalent to* f_M.

Note that if $\deg(f) = d$ then the vector space in (4) is one dimensional. In the case of f_M it can be canonically identified with $\mathbb{C}$ and then $\Delta(f_M) = 1$.

In the proof of the theorem we use vanishing of a holomorphic tensor obtained by means of a matrix valued generalization of the Schwartz derivative. It is a special case of a noncommutative generalization of the classical Schwartz derivative, where instead of complex matrices we can use an arbitrary associative (unital) algebra (e.g. quaternions or Clifford algebras relating it with conformal mappings) [14]. Independently, the matrix valued Schwartz derivative appeared in Control Theory, where it arises as an expression of the curvature of a curve in appropriate coordinates [1], [2], [19], [20]. A very general abstract approach to a noncommutative Schwarz derivative was proposed in [17].

2. Matrix valued Schwartz derivative.

Definition. Given a holomorphic matrix valued function

(11) $$\mathbb{C} \supset U \to M_d(\mathbb{C}),$$

(12) $$x \mapsto y = f(x),$$

whose derivative y' is an invertible matrix valued function we define the matrix valued quadratic differential

(13) $$\sigma(f) := (((y')^{-1}y'')' - \frac{1}{2}((y')^{-1}y'')^2)\mathrm{d}x^{\otimes 2},$$

which we call the *matrix valued Schwartz derivative* of f.

Lemma 1. σ *is invariant under* $\mathrm{GL}_2(\mathbb{C})$*-transformations*

(14) $$x \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot x = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}),$$

under $\mathrm{GL}_{2d}(\mathbb{C})$*-transformations*

(15) $$y \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot y = (Ay + B)(Cy + D)^{-1}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2d}(\mathbb{C}),$$

transforms as follows

(16) $$\sigma(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot f) = (Cf + D)\sigma(f)(Cf + D)^{-1},$$

and vanishes if and only if

(17) $$f(x) = (Ax + B)(Cx + D)^{-1}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2d}(\mathbb{C}).$$

Proof: The transformation rules can be checked by straightforward computation. Vanishing of $\sigma(f)$ is equivalent to the following system of ODE's

$$(18) \qquad\qquad\qquad\qquad y'' = 2y'z,$$

$$(19) \qquad\qquad\qquad\qquad z' = z^2.$$

On one hand, by (16) the map of the form (17) is a solution to the system (18)-(19), because the map $f(x) = x \cdot 1_d$ is such. On the other hand, the solution (17) with

$$(20) \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} y'_0 - y_0 z_0 & y_0 + y_0 z_0 x_0 - y'_0 x_0 \\ -z_0 & 1 + z_0 x_0 \end{pmatrix}$$

satisfies the initial condition $(x, y, y', z) = (x_0, y_0, y'_0, z_0)$. $\square$

Corollary 1. *Let* $f : \mathbb{P}^1 \to \mathbb{G}(d, 2d)$ *be a morphism such that*

$$(21) \qquad\qquad\qquad \deg(f) = d, \quad \Delta(f) \neq 0.$$

Then

1) the open subset $U := f^{-1}(M_d(\mathbb{C})) \subset \mathbb{P}^1$, *the pre-image of a big affine cell* $M_d(\mathbb{C}) \subset \mathbb{G}(d, 2d)$, *is contained in* $\mathbb{C} \subset \mathbb{P}^1$,

2) the first derivative of $f \mid_U : U \to M_d(\mathbb{C})$ *is an invertible matrix valued function,*

3) $\sigma(f \mid_U)$ *extends uniquely to an element*

$$(22) \qquad\qquad \sigma(f) \in \det_{\mathbb{C}}(\mathbb{C}^2)^{-2} \otimes_{\mathbb{C}} \mathrm{H}^0(\mathbb{P}^1, f^*(\mathcal{E}nd(S))(-4)).$$

Proof: Since $\deg(f) = d$ the sheaf $\det_{\mathbb{C}}(\mathbb{C}^d)^2 \otimes_{\mathbb{C}} (f^*(\det S)(d))^{-2}$ is a trivial line bundle. Therefore, if $\Delta(f) \neq 0$ then it is non-zero at every point, hence (3) is an isomorphism. In particular f is non-constant, what implies 1).

Every grassmannian $\mathbb{G}(k, n)$ admits an atlas by big affine cells $M_{k \times (n-k)}(\mathbb{C})$ with transition functions of the form

$$(23) \qquad y \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot y = (Ay + B)(Cy + D)^{-1}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}),$$

where transition functions for the tautological bundles S and Q are of the form

$$(24) \qquad (y, s) \mapsto ((Ay + B)(Cy + D)^{-1}, (Cy + D)s),$$

$$(25) \qquad (y, q) \mapsto ((Ay + B)(Cy + D)^{-1}, (A - (Ay + B)(Cy + D)^{-1}C)q).$$

Therefore, since

$$(26) \qquad ((Ay + B)(Cy + D)^{-1})' = (A - (Ay + B)(Cy + D)^{-1}C)y'(Cy + D)^{-1},$$

the morphism (2) restricted to U can be identified with the first derivative of $f \mid_U : U \to M_d(\mathbb{C})$, so the fact that it is an isomorphism implies 2). Finally, the transformation rules (13)-(15) imply 3). $\square$

Remark. One can assume above that $y^T = y$ (resp. $y^T = -y$) and restrict to transformations of the form

$$(27) \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2d}(\mathbb{C}) \quad (\text{resp. } \mathrm{O}_{2d}(\mathbb{C})),$$

to obtain the analogical atlas for lagrangian (resp. isotropic) grassmannians. One can also use the formula (12) to define the lagrangian (resp. isotropic) analog of the matrix valued Schwartz derivative. If we assume that y is a point of the subset $\mathcal{D}_d$ of the lagrangian grassmannian, consisting of positive definite lagrangian subspaces, i.e. $y^T = y$ and $I_d - y\bar{y}$ positive definite, and restrict to the transformations with

$$(28) \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \begin{pmatrix} iI_d & I_d \\ I_d & iI_d \end{pmatrix} \mathrm{Sp}_{2d}(\mathbb{Z}) \begin{pmatrix} iI_d & I_d \\ I_d & iI_d \end{pmatrix}^{-1} \subset \mathrm{Sp}_{2d}(\mathbb{C}),$$

we obtain the global structure of a quotient $\mathcal{A}_d = \mathrm{Sp}_{2d}(\mathbb{Z}) \setminus \mathcal{D}_d$ parameterizing isomorphism classes of complex principally polarized abelian varieties of dimension d [6]. In all such cases the restriction of the Euler sequence gives the short exact sequence

$$(29) \qquad 0 \to S \to \mathbb{C}^{2d} \otimes_{\mathbb{C}} \mathcal{O} \to S^{\vee} \to 0,$$

where S is the tautological bundle of lagrangian (resp. isotropic) subspaces in $\mathbb{C}^{2d}$ with the symplectic (resp. quadratic form) represented by the standard skew symmetric (resp. symmetric) matrix

$$(30) \qquad \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \left(\text{resp. } \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix} \right).$$

The tangent sheaf is then of the form $\mathcal{T} = \mathrm{Sym}^2(S^{\vee})$ (resp. $\mathcal{T} = \bigwedge^2(S^{\vee})$). By the Uniformization Theorem every Riemann surface C admits an atlas with transition functions of the form (13). Therefore our matrix valued Schwarz derivative is well defined for all holomorphic mappings f from C into usual grassmannians $\mathbb{G}(d, 2d)$, lagrangian or isotropic grassmannians, and Shimura varieties $\mathcal{A}_d$. Then

$$(31) \qquad \sigma(f) \in \mathrm{H}^0(C, f^*(\mathcal{E}nd(S))(2K_C)).$$

Corollary 2. *Let f be such as in Corollary 1. If $\sigma(f) = 0$ then f is $\mathrm{GL}_{2d}(\mathbb{C})$-equivalent to f_M.*

Proof: There exist big affine cells of the grassmannians $\mathbb{P}^1$ and $\mathbb{G}(d, 2d)$ for which the restriction of f_M takes the form

$$(32) \qquad x \mapsto x \cdot 1_d.$$

Since $\sigma(f) = 0$ we know by Lemma 1 that the restriction of f has to be of the form (16), which means that there exists an element $g \in \mathrm{GL}_{2d}(\mathbb{C})$ such that $f = g \circ f_M$. $\square$

3. Proof of the theorem. Using the Grothendieck splitting (5) we see that by (21) $\sigma(f)$ is an element of a vector space isomorphic to

$$(33) \qquad \bigoplus_{i,j=0}^{d} \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i - a_j - 4)).$$

If $\varpi(f) \leq 3$ then $a_i - a_j - 4 < 0$, for all i, j. Therefore the vector space (21) is zero, hence $\sigma(f) = 0$. By Corollary 2, this implies that f is $\mathrm{GL}_{2d}(\mathbb{C})$-equivalent to f_M. $\square$

4. Particular cases. Since $\varpi(f) \leq d$, the condition $\varpi(f) \leq 3$ is satisfied automatically for $d \leq 3$. For $d = 1$ or 2 our theorem reduces to the following simple and well known facts.

For $d = 1$ the Plücker embedding is an identity of $\mathbb{P}^1$, the conditions $\deg(f) = 1$, $\Delta(f) \neq 0$ mean that f is birational étale. The Morita curve f_M is the identity. Then our theorem is equivalent to the fact that f is a Möbius map.

For $d = 2$ the Plücker embedding $\mathbb{G}(2,4) \hookrightarrow \mathbb{P}^5 = \mathbb{P}(\bigwedge^2(\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^2))$ identifies the grassmannian with the Klein quadric

$$(34) \qquad z_{11,12}z_{21,22} - z_{11,21}z_{12,22} + z_{11,22}z_{12,21} = 0.$$

Then the image of the Morita curve is a closed embedding onto the intersection of the Klein quadric with the plane

$$(35) \qquad z_{11,21} = 0, \quad z_{12,22} = 0, \quad z_{11,22} + z_{12,21} = 0.$$

The conditions $\deg(f) = 2$, $\Delta(f) \neq 0$ are equivalent to the condition that f is a closed embedding onto a conic not contained in a plane entirely contained in the Klein quadric. Then our theorem is equivalent to the fact that all conics in $\mathbb{P}^5$ lying on the smooth quadric but not contained in a plane entirely contained in the quadric, are equivalent up to automorphisms of $\mathbb{P}^5$ preserving the quadric.

For $d = 3$ our theorem says about cubics lying on some 9-dimensional smooth intersection of quadrics in $\mathbb{P}^{19}$, but seems not to reduce to any simple classical fact.

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